Assignment 3

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Contents

1 Exercise 1

The problem in the matrix form is :

$$A_B = B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_N = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix}, x_N = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$c_B = \begin{bmatrix} 0 & 0 \end{bmatrix}, c_N = \begin{bmatrix} 5 & 3 & 4 \end{bmatrix}, x_B^* = \begin{bmatrix} 20 \\ 30 \end{bmatrix}$$

1.1 First iteration

1.1.1 Step 1 : Solve yB = c_B

$$y \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

1.1.2 Step 2 : Entering variable

$$ya < c_N? \Rightarrow \begin{bmatrix} 0 \times 2 + 0 \times 3 = 0 \\ 0 \times 1 + 0 \times 1 = 0 \\ 0 \times 1 + 0 \times 2 = 0 \end{bmatrix} < \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}?$$

We break the tie by choosing the lowest index : x_1 and so :

$$a = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

1.1.3 Step 3 : Solve B d = a

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow d = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

1.1.4 Step 4 : Leaving variable

$$x_B^* - td \geq 0 \Rightarrow \begin{bmatrix} 20 \\ 30 \end{bmatrix} - t \begin{bmatrix} 2 \\ 3 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow t = 10$$

We break the tie by choosing the lowest index : x_4

1.1.5 Step 5 : Updating

$$x_B^* = x_B^* - td = \begin{bmatrix} 10 \\ 30 - 10 \times 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$E_1 = B_1 = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}, A_N = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_1 \\ x_5 \end{bmatrix}, x_N = \begin{bmatrix} x_4 \\ x_2 \\ x_3 \end{bmatrix}$$

$$c_B = \begin{bmatrix} 5 & 0 \end{bmatrix}, c_N = \begin{bmatrix} 0 & 3 & 4 \end{bmatrix}$$

1.2 Second iteration

1.2.1 Step 1 : Solve yB = c_B

$$y\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \end{bmatrix} \Rightarrow y = \begin{bmatrix} \frac{5}{2} & \frac{-15}{2} \end{bmatrix}$$

1.2.2 Step 2 : Entering variable

$$ya < c_N? \Rightarrow \begin{bmatrix} \frac{5}{2} \\ \frac{5}{2} - \frac{15}{2} = -5 \\ \frac{5}{2} - 15 = -12,5 \end{bmatrix} < \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}?$$

We choose the maximum difference : x_3 and so :

$$a = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

1.2.3 Step 3 : Solve B d = a

$$\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} d = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow d = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

1.2.4 Step 4 : Leaving variable

$$x_B^* - td \geq 0 \Rightarrow \begin{bmatrix} 10 \\ 0 \end{bmatrix} - t\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow t = 0$$

We obtain the leaving variable : x_5

1.2.5 Step 5 : Updating

$$x_B^* = x_B^* - td = \begin{bmatrix} 10 - 0 \times \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}, A_N = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}, x_N = \begin{bmatrix} x_4 \\ x_2 \\ x_5 \end{bmatrix}$$

$$c_B = \begin{bmatrix} 5 & 4 \end{bmatrix}, c_N = \begin{bmatrix} 0 & 3 & 0 \end{bmatrix}$$

1.3 Third iteration

1.3.1 Step 1 : Solve yB = c_B

$$B_2 = E_1 E_2 \Rightarrow u\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 5 & 4 \end{bmatrix} \Rightarrow u = \begin{bmatrix} 5 & 3 \end{bmatrix}$$

$$y\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \end{bmatrix} \Rightarrow y = \begin{bmatrix} -2 & 3 \end{bmatrix}$$

1.3.2 Step 2 : Entering variable

$$ya < c_N? \Rightarrow \begin{bmatrix} -2 \\ -2+3=1 \\ 3 \end{bmatrix} < \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}?$$

We break the tie by choosing the lowest index : x_2 and so :

$$a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

1.3.3 Step 3 : Solve B d = a

$$\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} u = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow u = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} d = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \Rightarrow d = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

1.3.4 Step 4 : Leaving variable

$$x_B^* - td \geq 0 \Rightarrow \begin{bmatrix} 10 \\ 0 \end{bmatrix} - t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow t = 10$$

We obtain the leaving variable : x_1

1.3.5 Step 5 : Updating

$$x_B^* = x_B^* - td = \begin{bmatrix} 10 \\ 0 - 10(-1) = 10 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, A_N = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}, x_N = \begin{bmatrix} x_4 \\ x_1 \\ x_5 \end{bmatrix}$$

$$c_B = \begin{bmatrix} 3 & 4 \end{bmatrix}, c_N = \begin{bmatrix} 0 & 5 & 0 \end{bmatrix}$$

1.4 Fourth iteration

1.4.1 Step 1 : Solve yB = c_B

$$B_2 = E_1 E_2 E_3 \Rightarrow v \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \end{bmatrix} \Rightarrow v = \begin{bmatrix} 7 & 4 \end{bmatrix}$$

$$u \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 7 & 4 \end{bmatrix} \Rightarrow u = \begin{bmatrix} 7 & 1 \end{bmatrix}$$

$$y \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 1 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

1.4.2 Step 2 : Entering variable

$$ya < c_N? \Rightarrow \begin{bmatrix} & 2 \\ 2\times 2 + 3 \times 1 = & 5 \\ & 3 \end{bmatrix} < \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}?$$

There is no candidate, so the current solution is optimal :

$$\begin{cases} x_1^* = 0 \\ x_2^* = 10 \\ x_3^* = 10 \end{cases}$$

2 Exercise 2

2.1 Proof

Consider the following problem :

$$\begin{aligned} \text{minimize } Z &= by - cx \\ Ax &\leq b \\ A^T y &\geq c \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

as 2 problems :

$$\begin{aligned} \text{maximize } z &= cx \\ Ax &\leq b \\ x &\geq 0 \end{aligned}$$

and

$$\begin{aligned} \text{minimize } w &= by \\ A^T y &\geq c \\ y &\geq 0 \end{aligned}$$

You may have noticed that the second problem is the dual of the first one. Now let me remember you the Primal-Dual combinations :

Dual / Primal	Optimal	Infeasible	Unbounded
Optimal	Possible	Impossible	Impossible
Infeasible	Impossible	Possible	Possible
Unbounded	Impossible	Possible	Impossible

So if one of those problems is unbounded (respectively infeasible), the other is infeasible (respectively unbounded). We can conclude that our original problem is infeasible.

If we find an optimal solution for one of those problems, we can find a optimal solution for the other. So there exists y^* and x^* optimal solution that verify the Duality Theorem :

$$b.y^* = c.x^* \Rightarrow b.y^* - c.x^* = 0 \Rightarrow Z = 0$$

As we don't know b and c, the only way to verify this is to have :

$$y^* = 0 \text{ and } x^* = 0$$

2.2 Infeasible example

$$\begin{array}{rcl} \text{minimize } Z & = & -y_1 + 3y_2 - x_1 - x_2 \\ x_1 & \leq & -1 \\ x_2 & \leq & 3 \\ y_1 & \geq & -1 \\ y_2 & \geq & -1 \\ x & \geq & 0 \\ y & \geq & 0 \end{array}$$

2.3 Zero optimal solution example

$$\begin{array}{rcl} \text{minimize } Z & = & y_1 + 3y_2 - x_1 - x_2 \\ x_1 & \leq & 1 \\ x_2 & \leq & 3 \\ y_1 & \geq & -1 \\ y_2 & \geq & -1 \\ x & \geq & 0 \\ y & \geq & 0 \end{array}$$

3 Exercise 3

3.1 New optimal solution

To find the new optimal solution, we have to add the new constraint to the previous optimum dictionary :

$$\left\{ \begin{array}{rcl} x_1 & = & 10 - x_3 + x_4 - 2x_5 \\ x_2 & = & 20 - x_3 - x_4 + x_5 \\ x_6 & = & 25 - x_1 - x_2 - x_3 = -5 + x_3 + x_5 \\ Z & = & 170 - 2x_3 - x_4 - 4x_5 \end{array} \right.$$

Then, we have to apply the Dual Simplex Method: x_6 is the leaving as - 5 is negative. Then we have to consider x_3 and x_5 as possible entering variables. The ratio test gives us x_3 because $\frac{-2}{-1} < \frac{-4}{-1}$. The dictionary becomes :

$$\left\{ \begin{array}{rcl} x_1 & = & 5 + x_4 - x_5 + x_6 \\ x_2 & = & 15 - x_4 + 2x_5 + x_6 \\ x_3 & = & 5 - x_5 - x_6 \\ Z & = & 160 - x_4 - 2x_5 + 2x_6 \end{array} \right. \Rightarrow \left\{ \begin{array}{rcl} x_1 & = & 5 \\ x_2 & = & 15 \\ x_3 & = & 5 \\ Z & = & 160 \end{array} \right.$$

3.2 Maximum price

This question consists in how much do I lose due to the constraint : 170 - 160 = 10 (Millions dollars). That is the maximum price I have to pay to forget the new constraint and keep my actual schedule.